OC533. For $k \in \mathbb{Z}$, define the polynomial $F_k(x) = x^4 + 2(1-k)x^2 + (1+k)^2$. Find all values of k so that F_k is irreducible over $\mathbb{Z}[x]$ and reducible over $\mathbb{Z}_p[x]$ for all primes p.

Originally problem 4 from the 2018 Romania Math Olympiad, Final Round.

We received 5 solutions, of which 4 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

We claim that the result holds precisely when neither k nor $-k$ is a perfect square in $\mathbb{Z}$.

Note that if $\alpha^2 = k$ in a unitary commutative ring R, then

$$F_k(x) = \left(x^2 + 2\alpha x + (k+1)\right)\left(x^2 - 2\alpha x + (k+1)\right) \text{ in } R[x]. \quad (1)$$

Also, if $\beta^2 = -k$ in R, then

$$F_k(x) = \left(x^2 + 2\beta + (1-k)\right)\left(x^2 - 2\beta + (1-k)\right) \text{ in } R[x]. \quad (2)$$

Therefore if k or $-k$ is a perfect square in $\mathbb{Z}$, then F_k is reducible in $\mathbb{Z}[x]$. On the other hand if $-k$ is not a perfect square, then we claim F_k has no rational roots and hence no linear factor. If r were a rational root of $F_k(x)$, then r^2 would be a root of $x^2 + 2(1-k)x + (1+k)^2$, but the roots are $k - 1 \pm 2\sqrt{-k}$, and these are not rational.

If $\gamma^2 = -1$ in R, we have

$$F_k(x) = \left(x^2 + 2\gamma x - (1+k)\right)\left(x^2 - 2\gamma x - (1+k)\right) \text{ in } R[x]. \quad (3)$$

By unique factorization in $\mathbb{C}[x]$, equations (1), (2), and (3) are the only ways of factoring F_k into monic quadratics and the coefficient of x is not an integer in any of those factorizations if neither k nor $-k$ is a perfect square. Therefore F_k is irreducible in $\mathbb{Z}[x]$.

On the other hand, it is well known that in $\mathbb{Z}_p$ at least one of y, z, or yz must be a square. Therefore one of $k, -1$, or $-k$ must be a square and hence at least one of the factorizations above exists in $\mathbb{Z}_p[x]$.